Theorem I. *(Derivative of sum) If $\vec{u}(t)$ and $\vec{v}(t)$ be two differentiable functions of the scalar t, to show that*

$$\boxed{\frac{d}{dt}(\vec{u} \pm \vec{v}) = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}.}$$

That is, the differentiation of the sum of two differentiable vector functions of any scalar with respect to that scalar is equal to the sum of their derivatives with respect to the same scalar.

Proof. Let

$$\vec{w} = \vec{u} \pm \vec{v}. \qquad \ldots \quad (1)$$

Les $\delta \vec{u}$ and $\delta \vec{v}$ be the small increments in $\vec{u}$ and $\vec{v}$ respectively and $\delta \vec{w}$ be the corresponding small increment in $\vec{w}$.

Then $$\vec{w}+\delta \vec{w}=(\vec{u}+\delta \vec{u})\pm(\vec{v}+\delta \vec{v}). \qquad .. \qquad (2)$$

Subtracting the corresponding sides of (1) from (2), we get

$$\vec{w}+\delta \vec{w}-\vec{w}=(\vec{u}+\delta \vec{u})\pm(\vec{v}+\delta \vec{v})-(\vec{u}\pm\vec{v})\cdot$$

or $\delta \vec{w}=\delta \vec{u}\pm\delta \vec{v}$.

Dividing both sides by δt, we get

$$\frac{\delta \vec{w}}{\delta t}=\frac{\delta \vec{u}}{\delta t}\pm\frac{\delta \vec{v}}{\delta t}.$$

Now proceeding to limits $\delta t \to 0$, we get

$$\underset{\delta t\to 0}{\text{Lt}}\frac{\delta \vec{w}}{\delta t}=\underset{\delta t\to 0}{\text{Lt}}\frac{\delta \vec{u}}{\delta t}\pm\underset{\delta t\to 0}{\text{Lt}}\frac{\delta \vec{v}}{\delta t}$$

or $$\frac{d\vec{w}}{dt}=\frac{d\vec{u}}{dt}\pm\frac{d\vec{v}}{dt}.$$

Hence $$\frac{d}{dt}(\vec{u}\pm\vec{v})=\frac{d\vec{u}}{dt}\pm\frac{d\vec{v}}{dt}.$$

[**Note.** This rule for differentiation of sum of vectors may be extended to any number of vectors.]

That is, $$\frac{d}{dt}(\vec{u}\pm\vec{v}\pm\vec{h}\pm\ldots)=\frac{d\vec{u}}{dt}\pm\frac{d\vec{v}}{dt}\pm\frac{d\vec{h}}{dt}\pm\ldots$$

Theorem II. *(Derivative of product with a scalar) If $\vec{u}(t)$ be a differentiable vector function of the scalar t and $\varphi(t)$ the differentiable scalar function of t, to show that*

$$\boxed{\frac{d}{dt}(\vec{u}\,\varphi)=\frac{d\vec{u}}{dt}\varphi+\vec{u}\,\frac{d\varphi}{dt}.}$$

Proof. Let $$\vec{w}=\vec{u}\,\varphi. \qquad .. \qquad (1)$$

Let $\delta \vec{u}$ and $\delta\varphi$ be the small increments in $\vec{u}$ and φ respectively and $\delta \vec{w}$ be the corresponding small increment in $\vec{w}$.

Then $\vec{w}+\delta\vec{w}=(\vec{u}+\delta\vec{u})(\varphi+\delta\varphi)$

or $\vec{w}+\delta\vec{w}=\vec{u}\,\varphi+\delta\vec{u}\,\varphi+\vec{u}\,\delta\varphi+\delta\vec{u}\,\delta\varphi.$ .. (2)

Subtracting the corresponding sides of (1) from (2), we get

$$\vec{w}+\delta\vec{w}-\vec{w}=\vec{u}\,\varphi+\delta\vec{u}\,\varphi+\vec{u}\,\delta\varphi+\delta\vec{u}\,\delta\varphi-\vec{u}\,\varphi$$

or $\delta\vec{w}=\delta\vec{u}\,\varphi+\vec{u}\,\delta\varphi+\delta\vec{u}\,\delta\varphi.$

Dividing both sides by δt, we get

$$\frac{\delta\vec{w}}{\delta t}=\frac{\delta\vec{u}}{\delta t}\varphi+\vec{u}\,\frac{\delta\varphi}{\delta t}+\frac{\delta\vec{u}}{\delta t}\cdot\frac{\delta\varphi}{\delta t}.\delta t.$$

Now proceeding to limits as $\delta t\to 0$, we get

$$\operatorname*{Lt}_{\delta t\to 0}\frac{\delta\vec{w}}{\delta t}=\operatorname*{Lt}_{\delta t\to 0}\frac{\delta\vec{u}}{\delta t}\varphi+\vec{u}\operatorname*{Lt}_{\delta t\to 0}\frac{\delta\varphi}{\delta t}$$

$$+\operatorname*{Lt}_{\delta t\to 0}\frac{\delta\vec{u}}{\delta t}.\operatorname*{Lt}_{\delta t\to 0}\frac{\delta\varphi}{\delta t}.\operatorname*{Lt}_{\delta t\to 0}\delta t$$

or $$\frac{d\vec{w}}{dt}=\frac{d\vec{u}}{dt}\varphi+\vec{u}\,\frac{d\varphi}{dt}+\frac{d\vec{u}}{dt}\cdot\frac{d\varphi}{dt}.0.$$

Hence $$\frac{d}{dt}(\vec{u}\,\varphi)=\frac{d\vec{u}}{dt}\varphi+\vec{u}\,\frac{d\varphi}{dt}.$$